RESTRICTING THE ROST INVARIANT TO THE CENTER

SKIP GARIBALDI AND ANNE QUÉGUINER-MATHIEU

Abstract. For simple simply connected algebraic groups of classical type, Merkurjev, Parimala, and Tignol gave a formula for the restriction of the Rost invariant to torsors induced from the center of the group. We complete their results by proving formulas for exceptional groups. Our method is somewhat different and recovers also their formula for classical groups.

1. Introduction

In the 1990's, M. Rost proved that the group of degree 3 normalized invariants of an absolutely simple simply connected algebraic group G over k — that is, the group of natural transformations of the Galois cohomology functors

$$H^1(\star, G) \to H^3(\star, \mathbb{Q}/\mathbb{Z}(2))$$

— is a cyclic group with a canonical generator. This canonical generator is known as the *Rost invariant*. Roughly speaking, it is the "first" nonzero invariant, in that there are no non-zero normalized invariants $H^1(\star, G) \to H^d(\star, \mathbb{Q}/\mathbb{Z}(d-1))$ for $d < 3$ [KMRT, §31].

In general, there is no explicit formula for computing the Rost invariant. However, in [MPT], Merkurjev, Parimala and Tignol gave a nice description — for G of classical type — of the restriction of the Rost invariant to $H^1(\star, Z)$, where Z is the center of G. The purpose of this paper is to complete their results by providing an analogous description for exceptional groups (see Th. 2.3 for a precise statement for all types of groups).

Part of our proof is borrowed from [MPT]. Precisely, their Corollaries 1.2 to 1.6, which are stated for a general cycle module, show in our context that every group invariant

$$H^1(\star, Z) \to H^3(\star, \mathbb{Q}/\mathbb{Z}(2))$$

— i.e., every invariant such that the map $H^1(K, Z) \to H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ is a group homomorphism for every extension K/k — is given by some cup-product with a class $t \in H^2(k, Z)$, see Prop. 2.1 and §3 below. (Note that this class t does depend on a choice regarding the cup product, see Remark 2.5(iii).)

It remains to compute the class $t_{R,G} \in H^2(k, Z)$ associated with composition

$$H^1(\star, Z) \longrightarrow H^1(\star, G) \xrightarrow{r_G} H^3(\star, \mathbb{Q}/\mathbb{Z}(2)), \tag{1.1}$$

where the first map is induced by the inclusion of Z in G and r_G is the Rost invariant of G. Note that, even though it is not obvious from its definition, this composition actually is a group invariant, see [Ga 01b, 7.1] or [MPT, Cor. 1.8]. The Rost invariant is canonically determined; nevertheless, practically speaking, one

Date: February 2, 2008.

only knows the group it generates. Consequently, we will determine the subgroup $\langle t_{R,G}\rangle$ of $H^2(k,Z)$ generated by $t_{R,G}$.

For any semisimple group G, one has the Tits class t_G, an element of the group $H^2(k,Z)$. It follows from [MPT] and from the present paper that *either $t_{R,G}$ is zero or $t_{R,G}$ and t_G generate the same subgroup of $H^2(k,Z)$*, for a well-chosen cup product. Note that our argument for determining $\langle t_{R,G}\rangle$ is quite different from [MPT], in which they use concrete interpretations of the classical groups. Here, we first reduce to groups whose Tits index satisfies a certain condition (see (4.1)), using an injectivity result which follows from [MT, Th. B]. Second, we prove the result for those particular groups, by reducing to groups of inner type A for which the Rost invariant has a concrete description. This method applies to all absolutely simple groups except those of outer type A, and therefore can be used to recover the results of [MPT] on classical groups (except for their Th. 1.10). We give the details for types B, C, and D_{even}, as well as for the exceptional groups.

2. Statement of results

Throughout the paper, G denotes an absolutely simple simply connected algebraic group over a field k. We assume (except in §6) that *the characteristic of k does not divide the exponent n of the center of G*. That is, the characteristic is not 2 for G of type B, C, D, E_7; the characteristic is not 3 for G of type E_6; and the characteristic does not divide $\ell+1$ for G of type A_ℓ. This guarantees that the (scheme-theoretic) center Z of G is smooth. (See Remark 4.4 for comments on the characteristic hypothesis.)

The group $H^d(k,\mathbb{Q}/\mathbb{Z}(d-1))$ is as defined in [M 03, App. A]; otherwise the notation $H^d(k,A)$ stands for the Galois cohomology group $H^d(\Gamma_k, A(k_{\mathrm{sep}}))$, where Γ_k denotes the Galois group of a separable closure k_{sep} of k over k and A is a smooth algebraic group. Our hypothesis on the characteristic of k implies that the n-torsion of $H^d(k,\mathbb{Q}/\mathbb{Z}(d-1))$ is naturally identified with $H^d(k,\boldsymbol{\mu}_n^{\otimes(d-1)})$ for $d=2,3$. When G is not of type A_ℓ with $\ell\geq 3$ — e.g., in all cases studied in detail below — n is 2 or 3, hence $\boldsymbol{\mu}_n^{\otimes 2}$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ (see [KMRT, Ex. 11, p. 444]).

For any functor $\mathcal{F}$ from the category of field extensions of k to the category of groups, we let $\mathrm{Inv}^3(\mathcal{F})$ be the collection of group invariants of $\mathcal{F}$ with values in $H^3(\star,\mathbb{Q}/\mathbb{Z}(2))$; it is an abelian group. If T is a quasi-trivial torus, i.e., $T=R_{E/k}(\mathbb{G}_m)$ for some étale algebra E/k, it follows from Th. 1.1 of [MPT] that $H^2(k,T)$ is isomorphic to $\mathrm{Inv}^3(T)$. Precisely, the invariant $\alpha^E(X)$ associated to a cohomology class $X\in H^2(k,T)$ is defined by $\alpha^E(X)(y)=N_{E_K/K}(y\cdot X_K)$ for every field extension K/k and $y\in T(K)=E_K^\times$. The cup product appearing in this formula is given by the module structure on $H^\star(\mathbb{Q}/\mathbb{Z}(-1))$ over the Milnor K-ring (see [M 03, Appendix A] for a definition). Using this, we may prove the following:

Proposition 2.1. *The groups $\mathrm{Inv}^3(H^1(\star,Z))$ and $H^2(k,Z)$ are (non-canonically) isomorphic.*

While proving this proposition in section 3, we will exhibit particular such isomorphisms, which admit the following explicit description: for any cohomology class $X\in H^2(k,Z)$, the corresponding invariant is given by

$$y\in H^1(K,Z)\mapsto y\cdot X_K\in H^3(K,\mathbb{Q}/\mathbb{Z}(2)).$$

Since it is a group invariant, its image is contained in the n-torsion, $H^3(K, \boldsymbol{\mu}_n^{\otimes 2})$, of $H^3(K, \mathbb{Q}/\mathbb{Z}(2))$. Further, the cup product is induced by a bilinear form $Z(k_{\mathrm{sep}}) \times Z(k_{\mathrm{sep}}) \to \boldsymbol{\mu}_n^{\otimes 2}$, which has to be specified.

We denote by $\boldsymbol{\mu}_n$ the algebraic group of nth roots of unity and by $\boldsymbol{\mu}_{n[E]}$ the kernel of the norm map $N_{E/k} : R_{E/k}(\boldsymbol{\mu}_n) \to \boldsymbol{\mu}_n$, for any quadratic étale algebra E/k. As recalled in [MPT], one deduces from the classification of absolutely simple simply connected groups that if G is classical, then Z is one of the following groups: $\boldsymbol{\mu}_n$, $R_{E/k}(\boldsymbol{\mu}_2)$, and $\boldsymbol{\mu}_{n[E]}$, where E is quadratic étale over k. If we wish to consider also exceptional groups, we need to add the centers of trialitarian D_4 groups, which are isomorphic to the kernel, now denoted by $R^1_{E/k}(\boldsymbol{\mu}_2)$, of the norm map $N_{E/k} : R_{E/k}(\boldsymbol{\mu}_2) \to \boldsymbol{\mu}_2$, where E is cubic étale over k.

In most cases, namely when Z is $\boldsymbol{\mu}_n$ or $\boldsymbol{\mu}_{n[E]}$, there is a natural bilinear map

$$Z(k_{\mathrm{sep}}) \times Z(k_{\mathrm{sep}}) \to \boldsymbol{\mu}_n^{\otimes 2}.$$

The only groups for which the cup-product has to be defined carefully are groups of type D_ℓ with ℓ even. Their center is—over k_{sep}—isomorphic to $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$. Rather than fix an identification between these two groups, we note that the fundamental weights of the root system give characters $\omega_1, \omega_{\ell-1}, \omega_\ell$ whose restriction to Z are the three nonzero homomorphisms $Z \to \boldsymbol{\mu}_2$. We consider the cup product induced by the bilinear map with values in $\boldsymbol{\mu}_2^{\otimes 2} = \mathbb{Z}/2\mathbb{Z}$:

$$(x,y) \mapsto \begin{cases} \omega_{\ell-1}(x) \otimes \omega_\ell(y) + \omega_\ell(x) \otimes \omega_{\ell-1}(y) & \text{if } \ell \equiv 0 \pmod 4, \\ \omega_{\ell-1}(x) \otimes \omega_{\ell-1}(y) + \omega_\ell(x) \otimes \omega_\ell(y) & \text{if } \ell \equiv 2 \pmod 4. \end{cases} \tag{2.2}$$

Since $\omega_1 + \omega_{\ell-1} + \omega_\ell = 0$ as characters, when $\ell \equiv 0 \pmod 4$, this cup product can be rewritten as

$$(x,y) \mapsto \omega_1(x) \otimes \omega_1(y) + \omega_{\ell-1}(x) \otimes \omega_{\ell-1}(y) + \omega_\ell(x) \otimes \omega_\ell(y).$$

With this in hand, we may summarize the results of [MPT] and the present paper in the following theorem:[1]

Main Theorem 2.3. *If G is of type A, C_ℓ (ℓ odd), D, E_6 or E_7, then the composition* (1.1) *and the cup product with the Tits class of G generate the same subgroup of* $\mathrm{Inv}^3(H^1(\star, Z))$. *Otherwise the composition* (1.1) *is zero.*

Example 2.4. For groups of inner type A — that is, when G is $SL(A)$ for some central simple algebra A over k — the theorem asserts that (1.1) and the cup product with the Brauer class $[A]$ of A generate the same subgroup of $\mathrm{Inv}^3(H^1(\star, Z))$. This follows from the explicit description of the Rost invariant given in [M 03, p. 138]. This example is a starting point for proving the theorem.

The new cases in the theorem are the groups of type E_6, E_7, and trialitarian D_4, treated below in sections 11, 9, and 12 respectively. The exceptional groups of type E_8, F_4, and G_2 all have trivial center, so (1.1) is automatically zero for those groups.

Remark 2.5. (i) *If the Tits class is zero, then a twisting argument shows directly that the composition* (1.1) *is zero, regardless of the type of G.* Indeed, since the

[1] There is a typo in Th. 1.13 of [MPT], which addresses the C_ℓ case: the words “odd” and “even” should be interchanged. For keeping the C_{odd} and C_{even} cases straight, we find it helpful to recall that $B_2 = C_2$, cf. §10 below.

Tits class is zero, there is a simply connected quasi-split group G^q such that we may identify G with G^q twisted by a 1-cocycle $\alpha \in Z^1(k, G^q)$. We write Z^q for the center of G^q; the identification of G with G^q_α identifies Z with Z^q; we write i and i^q for the inclusions of Z and Z^q in G and G^q. The diagram

$$\begin{tikzcd} H^1(k,Z) \arrow[r, "i_\star"] & H^1(k,G) \arrow[r, "r_G"] \arrow[d, "\tau_\alpha"', "\cong"] & H^3(k,\mathbb{Q}/\mathbb{Z}(2)) \arrow[d, "?+r_{G^q}(\alpha)"] \\ H^1(k,Z^q) \arrow[r, "i^q_\star"] & H^1(k,G^q) \arrow[r, "r_{G^q}"] & H^3(k,\mathbb{Q}/\mathbb{Z}(2)) \end{tikzcd}$$

where τ_α is the twisting isomorphism, commutes by [Gi 00, p. 76, Lemma 7] or [MPT, Cor. 1.8]. Fix $\zeta \in Z^1(k, Z)$. The image of $i_\star\zeta$ under τ_α is the class in $H^1(k, G^q)$ of the 1-cocycle $\sigma \mapsto \zeta_\sigma\alpha_\sigma$ for $\sigma \in \Gamma_k$. But this is just $\zeta \cdot \alpha \in H^1(k, G^q)$, where the $\cdot$ represents the usual action of $H^1(k, Z^q)$ on $H^1(k, G^q)$. So the image of $i_\star\zeta$ going counterclockwise around the diagram is

$$r_{G^q}(\zeta \cdot \alpha) = r_{G^q}(i^q_\star\zeta) + r_{G^q}(\alpha)$$

by [Ga 01b, 7.1]. But G^q is quasi-split, so it has a quasi-trivial maximal torus T^q that necessarily contains Z^q, hence $i^q_\star$ factors through the zero group $H^1(k, T^q)$. We conclude that the image of $i_\star\zeta$ in the lower right corner is $r_{G^q}(\alpha)$.

On the other hand, going clockwise around the diagram, we find that the image of $i_\star\zeta$ in the lower right corner is $r_G(i_\star\zeta) + r_{G^q}(\alpha)$. The commutativity of the diagram proves the claim.

(ii) As opposed to (i), note: For groups of type B_ℓ, i.e., for $G = \mathrm{Spin}(q)$ for some $(2\ell+1)$-dimensional quadratic form q over k, the composition (1.1) is always zero, while the Tits class of G, which is the Brauer class of the even Clifford algebra of q, can be non-zero.

(iii) In the D_ℓ case with ℓ even, the bilinear form we use to compute the cup product does depend on whether $\ell/2$ is even or odd. If we keep the same cup product in both cases, then composition (1.1) coincides alternately with the cup product with t_G and its conjugate.

(iv) The bilinear maps in (2.2) can be viewed as symmetric bilinear forms on a vector space of dimension 2 over the field with two elements. For $\ell \equiv 0 \pmod 4$, it is the wedge product, equivalently, a hyperbolic form. For $\ell \equiv 2 \pmod 4$, it is the unique (up to isomorphism) metabolic form that is not hyperbolic.

3. Invariants of $H^1(\star, Z)$

The purpose of this section is to prove Proposition 2.1. In order to use the description of invariants of quasi-trivial tori recalled above, we need to embed Z in such a torus. More precisely, we are going to consider exact sequences

$$1 \longrightarrow Z \xrightarrow{j} T \xrightarrow{g} T \longrightarrow 1,$$

where T is a quasi-trivial torus, i.e., $T = R_{E/k}(\mathbb{G}_m)$ for some étale k-algebra E. We call such an exact sequence *admissible* if the map g satisfies the following equation:

$$N_{E_K/K}(g_K(y)\cdot X_K) = N_{E_K/K}(y\cdot H^2(g)(X)_K), \tag{3.1}$$

for every $X \in H^2(k, T)$, field extension K/k, and $y \in T(K) = E_K^\times$.

Given an admisible exact sequence, we claim it induces an isomorphism between $\mathrm{Inv}^3(H^1(\star,Z))$ and $H^2(k,Z)$, as in Prop. 2.1. Indeed we can apply [MPT, Lemma 3.1]: Since the sequence

$$T(K) \xrightarrow{g_K} T(K) \xrightarrow{\varphi} H^1(K,Z) \longrightarrow 1.$$

is exact for every extension K/k, the top row of the diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Inv}^3(H^1(\star,Z)) \arrow[r, "\varphi^\star"] & \mathrm{Inv}^3(T) \arrow[r, "g^\star"] & \mathrm{Inv}^3(T) \\ 1 \arrow[r] & H^2(k,Z) \arrow[r, "H^2(j)"] \arrow[u, dashed, "\beta"] & H^2(k,T) \arrow[r, "H^2(g)"] \arrow[u, "\alpha^E", "\cong"'] & H^2(k,T) \arrow[u, "\alpha^E", "\cong"'] \end{tikzcd} \tag{3.2}$$

is also exact, where α^E is the isomorphism described in §2. Since T is quasi-trivial, the group $H^1(k,T)$ is zero and the bottom row of the diagram is also exact. An easy computation shows that under condition (3.1) the right-hand box is commutative. Since α^E is an isomorphism, this enable us to identify the kernels of both lines, producing an isomorphism β as in the diagram.

To prove Prop. 2.1, it only remains to show that there exists an admissible exact sequence for the center Z of a given G. This is done in [MPT] in sections 3.1 to 3.4 when Z is isomorphic to $\boldsymbol{\mu}_n$, $R_{E/k}(\boldsymbol{\mu}_n)$, or $\boldsymbol{\mu}_{n[E]}$ (see also sequences (3), (6), and (8)). The only remaining case is the center of a trialitarian D_4. For such a group, there exists a cubic field extension E/k and an embedding $j : Z \to R_{E/k}(\mathbb{G}_m)$ which identifies Z with the kernel $R^1_{E/k}(\mathbb{G}_m)$ of the norm map. Over k_{sep}, it is given by $j_{\mathrm{sep}}(x) = (\omega_1(x), \omega_{\ell-1}(x), \omega_\ell(x)) \in (k_{\mathrm{sep}}^\times)^3$.

Let us define $g\colon R_{E/k}(\mathbb{G}_m) \to R_{E/k}(\mathbb{G}_m)$ by $g_K(x) = N_{E_K/K}(x)\,x^{-2}$, for any field K/k and any $x \in R_{E/k}(\mathbb{G}_m)(K) = E_K^\times$, and consider the sequence

$$1 \longrightarrow Z \xrightarrow{j} R_{E/k}(\mathbb{G}_m) \xrightarrow{g} R_{E/k}(\mathbb{G}_m) \longrightarrow 1 \tag{3.3}$$

Lemma 3.4. *Sequence 3.3 is exact and admissible.*

Proof. An element x in the kernel of g_K satisfies $N_{E_K/K}(x) = x^2$. Since E is cubic, taking the norm on both sides gives $1 = N_{E_K/K}(x) = x^2$, and this proves $x \in Z(K)$. The reader may easily check surjectivity; admissibility condition 3.1 follows from the projection formula, as in [MPT, 3.3]. □

Proposition 2.1 is now proved.

Note that, if you allow E/k to be a cubic étale algebra rather than a field extension, then the sequence (3.3) actually is an admissible exact sequence for any group of type D_ℓ with $\ell \equiv 0 \mod 4$. As we will see in the next paragraph, the corresponding isomorphism between $\mathrm{Inv}^3(H^1(\star,Z))$ and $H^2(k,Z)$ is not the same as the one induced by sequence (3) of [MPT] (see also Remark 2.5(iii)).

Hence we have to specify which exact sequence we use: from now on, we will always consider the isomorphism induced by the following admissible exact sequence, depending on Z and G:

- If $Z = \boldsymbol{\mu}_n$, we take

$$1 \longrightarrow \boldsymbol{\mu}_n \longrightarrow \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \longrightarrow 1.$$

- If $Z = \boldsymbol{\mu}_{n[E]}$ with n odd (respectively even), we take sequence (6) (resp., (8)) of [MPT]

- If G is of type D_ℓ with $\ell \equiv 2 \mod 4$, we take sequence (3) of [MPT].
- If G is of type D_ℓ with $\ell \equiv 0 \mod 4$, we take sequence (3.3).

In each of these cases, the isomorphism β from (3.2) is given by

$$\beta(t)(x) = x \cdot t_K, \tag{3.5}$$

where the cup product is induced by the bilinear map specified in § 2. (This is the description of the isomorphism β given just after the statement of Prop. 2.1.) This fact can be proved in each case by some explicit computation, going through the diagram. Let us do it for groups of type D_ℓ with $\ell \equiv 0 \mod 4$, and sequence (3.3).

For any $t \in H^2(k, Z)$, the invariant $\beta(t)$ is characterized by

$$\beta(t)(\varphi(x)) = \alpha^E(H^2(j)(t))(x) = N_{E_K/K}(x \cdot H^2(j)(t_K)) \tag{3.6}$$

for any $x \in T(K)$. We want to prove this is equal to $\varphi(x) \cdot t_K$, where the cup product is induced by the bilinear map

$$(x, y) \mapsto \omega_1(x) \otimes \omega_1(y) + \omega_{\ell-1}(x) \otimes \omega_{\ell-1}(y) + \omega_\ell(x) \otimes \omega_\ell(y).$$

Since Z has exponent 2, the map j factors as

$$Z \xrightarrow{j_0} R_{E/k}(\boldsymbol{\mu}_2) \longrightarrow R_{E/k}(\mathbb{G}_m),$$

and $j_{0\,\mathrm{sep}}$, as j_{sep}, coincides with the triple $(\omega_1, \omega_{\ell-1}, \omega_\ell)$. Hence, we have

$$\varphi(x) \cdot t_K = N_{E_K/K}(H^1(j_0)(\varphi(x)) \cdot H^2(j_0)(t_K)). \tag{3.7}$$

To compare (3.6) and (3.7), we use the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & Z \arrow[r, "j"] \arrow[d, "j_0"] & R_{E/k}(\mathbb{G}_m) \arrow[r, "g"] \arrow[d, equal] & R_{E/k}(\mathbb{G}_m) \arrow[r] \arrow[d, "h"] & 1 \\
1 \arrow[r] & R_{E/k}(\boldsymbol{\mu}_2) \arrow[r] & R_{E/k}(\mathbb{G}_m) \arrow[r, "2"] & R_{E/k}(\mathbb{G}_m) \arrow[r] & 1
\end{tikzcd}$$

where the map h is defined by $h_K(x) = N_{E_K/K}(x)x^{-1} = xg_K(x)$ for any $x \in R_{E/k}(\mathbb{G}_m)(K) = E_K^\times$. It induces a commutative square

$$\begin{tikzcd}
E_K^\times \arrow[r, "\varphi"] \arrow[d, "h_K"] & H^1(K, Z) \arrow[d, "H^1(j_0)"] \\
E_K^\times \arrow[r] & H^1(K, R_{E/k}(\boldsymbol{\mu}_2))
\end{tikzcd}$$

Hence $H^1(j_0)(\varphi(x)) = (h_K(x))_2 = (x)_2 + (g_K(x))_2$, and

$$\begin{aligned}
&N_{E_K/K}(H^1(j_0)(\varphi(x)) \cdot H^2(j_0)(t_K)) \\
&\qquad = N_{E_K/K}((x)_2 \cdot H^2(j_0)(t_K)) + N_{E_K/K}((g_K(x))_2 \cdot H^2(j_0)(t_K)) \\
&\qquad = N_{E_K/K}(x \cdot H^2(j)(t_K)) + N_{E_K/K}(g_K(x) \cdot H^2(j)(t_K)),
\end{aligned}$$

where the cup product in the last line is again given by the module structure on $H^\star(\mathbb{Q}/\mathbb{Z}(-1))$ over the Milnor K-ring (see [M 03, Appendix A]). By the admissibility condition (3.1), the second term in the sum is $N_{E_K/K}(x \cdot H^2(g) \circ H^2(j)(t_K)) = 0$, so the expressions in (3.6) and (3.7) are equal. This finishes the proof of (3.5).

4. Reduction to groups having a particular Tits index

For each group G, we let $t_{R,G}$ be the class in $H^2(k,Z)$ corresponding to the composition (1.1) under the isomorphism specified in the previous section. The main theorem asserts that this class is zero or generates the same subgroup of $H^2(k,Z)$ as the Tits class t_G, depending on the type of the group G we started with. A case by case proof will be given in sections 7 to 12. First, we prove some general facts, on which our strategy is based.

Given G, we fix a maximal k-split torus S, a maximal k-torus T containing S, and a set of simple roots Δ of G with respect to T. Recall the *Tits index of G* as defined in [Ti 66]. It is the data of the Dynkin diagram of G together with the action of the Galois group $\mathrm{Gal}(k_{\mathrm{sep}}/k)$ on Δ via the $*$-action, and the set Δ_0 of those $\alpha \in \Delta$ that vanish on S. Notations for Δ_0 vary, but following [Ti 66], we circle a vertex if it belongs to $\Delta \setminus \Delta_0$, and circle together vertices that are in the same Galois orbit.

Let Δ_r be the subset of Δ consisting of those simple roots such that the corresponding fundamental weight belongs to the root lattice. We consider the following condition on G:

(4.1) *No $\alpha \in \Delta_r$ vanishes on S.*

In terms of Tits indices, this amounts to

(4.2) *Every vertex in Δ_r is circled in the Tits index of G.*

Dynkin diagrams for E_7, E_6, and D_{even} with the vertices in Δ_r circled can be found in sections 9, 11, and 12 below.

4.3. We now give a method that reduces the proof of the main theorem 2.3 to the case where G satisfies condition (4.2). Let G be a simply connected absolutely almost simple group over k and further suppose that G has inner type, i.e., that the absolute Galois group acts trivially on the Dynkin diagram of G. Because G has inner type, the center Z of G is isomorphic to $\boldsymbol{\mu}_n$ or $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$, and in §3 we fixed an injection $j\colon Z \to \mathbb{G}_m^{\times s}$ for $s \leq 3$.

Consulting the tables in [Bou], we observe that every element of Δ_r is fixed by every automorphism of the Dynkin diagram, hence is fixed by the $*$-action. It follows that the variety of parabolic subgroups of G_{sep} of type $\Delta \setminus \Delta_r$ (in the notation of [BT, 4.2, 5.12] or [MT, p. 33]) is defined over k, see [BT, 5.24] or [BS, 8.4]. This variety has a point over its function field F, hence G satisfies condition (4.2) *over F*. We have a commutative diagram with exact rows

$$\begin{array}{ccccc}
1 & \longrightarrow & H^2(k,Z) & \xrightarrow{H^2(j)} & \oplus^s H^2(k,\mathbb{G}_m) \\
 & & \downarrow & & \downarrow \\
1 & \longrightarrow & H^2(F,Z) & \xrightarrow{H^2(j)} & \oplus^s H^2(F,\mathbb{G}_m).
\end{array}$$

The kernel of the restriction map $H^2(k,\mathbb{G}_m) \to H^2(F,\mathbb{G}_m)$ has been computed by Merkurjev and Tignol in [MT, Th. B]. Let Λ/Λ_r be the quotient of the weight lattice of G by its root lattice. The kernel of the restriction map is generated by the Tits algebras associated with the classes in Λ/Λ_r of the fundamental weights corresponding to those simple roots which belong to Δ_r. But we have precisely chosen Δ_r so that these classes are zero. Hence, the restriction map $H^2(k,\mathbb{G}_m) \to$

$H^2(F, \mathbb{G}_m)$ is injective. By commutativity of the diagram, the map $H^2(k, Z) \to H^2(F, Z)$ is injective.

Let $t_{R,G}$ and the Tits class $t_G \in H^2(k, Z)$ be as defined in the introduction. The isomorphism β between $H^2(k, Z)$ and $\mathrm{Inv}^3(H^1(\star, Z))$ from §3 is functorial in k, i.e., the image of $t_{R,G}$ in $H^2(F, Z)$ is the element corresponding to the composition (1.1) over F. Therefore, if the composition (1.1) is zero over F, then $t_{R,G}$ is killed by F, and $t_{R,G}$ is zero in $H^2(k, Z)$, i.e., the composition (1.1) is zero over k. Alternatively, if the restrictions of $t_{R,G}$ and t_G generate the same subgroup of $H^2(F, Z)$, then $t_{R,G}$ and t_G generate the same subgroup of $H^2(k, Z)$.

In this way, we can reduce the proof of the main theorem for groups of inner type to the case of groups of inner type satisfying (4.2). This reduction can be generalized to treat also groups of outer type, but we will not use that, so we omit it.

Remark 4.4. (i) For groups of type A, Δ_r is the empty set, so (4.2) vacuously holds, the function field F is k itself, and this reduction is useless.

(ii) Our global hypothesis excluding fields of certain characteristics arises from this argument, i.e., from 4.3. The hypothesis on the characteristic is needed for the identification of $H^2(k, Z)$ with $\mathrm{Inv}^3(H^1(\star, Z))$ from Prop. 2.1, because the proof of that proposition used [MPT, Th. 1.1] to provide the isomorphism α^E. In turn, the proof of the result from [MPT] needs that the n-torsion in $H^3(k, \mathbb{Q}/\mathbb{Z}(2))$ is a cycle module, in particular, one should be able to take residues relative to discrete valuations on k. This is only known for n not divisible by $\operatorname{char} k$.

With small changes, our proof for groups of inner type satisfying (4.2) holds without any restriction on the characteristic of k.

5. A semisimple subgroup

In this section, we assume that the group G has Tits index satisfying condition (4.2). We produce a semisimple simply connected subgroup G' of G that contains the center of G and describe how to compute the Rost invariant of some elements of G using G'.

5.1. **Description of G by generators and relations.** Recall from [St] that — over k_{sep} — G is generated by the images of homomorphisms

$$x_\alpha \colon \mathbb{G}_a \to G$$

as α varies over the set of roots Φ of G with respect to T. Write Λ_r and Λ for the root and weight lattices of G with respect to T. Since G is simply connected, Λ is identified with the character group $T^\star = \mathrm{Hom}_{k_{\mathrm{sep}}}(T_{\mathrm{sep}}, \mathbb{G}_m)$ and Λ_r with the character group $\overline{T}^\star$ of the image $\overline{T}$ of T in G/Z.

Write $T_\star$ for the Γ_k-module of *loops* or cocharacters, i.e., k_{sep}-homomorphisms $\mathbb{G}_m \to T_{\mathrm{sep}}$. There is a natural pairing $T^\star \times T_\star \to \mathbb{Z}$ that enables us to identify $T_\star$ with the dual $\mathrm{Hom}(\Lambda, \mathbb{Z})$ of $T^\star$, i.e., with the lattice of co-roots, denoted by $\Lambda_r^\vee$ (see [Bou, VI.1.1, Prop. 2]). We fix a set of simple roots $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$ of Φ; the coroots $\check\alpha_1, \check\alpha_2, \dots, \check\alpha_\ell$ are a set of simple roots of the dual root system $\Phi^\vee$ [Bou, VI.1.5, Rem. 5]. Given an element $\check\alpha := \sum c_i \check\alpha_i$ in $\Lambda_r^\vee$, the corresponding loop $\mathbb{G}_m \to T_{\mathrm{sep}}$ is

$$t \mapsto \prod h_i(t^{c_i}), \tag{5.2}$$

where $h_i\colon \mathbb{G}_m \to T_{\mathrm{sep}}$ is the loop corresponding to $\check{\alpha}_i$. The map $\prod h_i\colon \mathbb{G}_m^{\times \ell} \to T_{\mathrm{sep}}$ is an isomorphism [St, p. 44, Cor. (a)]. We have the identity:

$$h_{\check{\alpha}}(t) x_\beta(u) h_{\check{\alpha}}(t)^{-1} = x_\beta(t^{(\beta,\check{\alpha})}u) \qquad (\beta \in \Phi, \check{\alpha} \in \Phi^\vee). \tag{5.3}$$

5.4. Definition of G'. Assume that G satisfies (4.2). Write $\Lambda^\vee$ for the weight lattice of the dual root system; it has a basis, dual to the basis Δ of Λ_r, and we write $\check{\omega}_j$ for the "co-weight" such that $(\check{\omega}_j, \alpha_i) = \delta_{ij}$ (Kronecker delta). Write $\overline{M}$ for the sublattice of $\Lambda^\vee$ generated by the $\check{\omega}_j$ for $\alpha_j \in \Delta_r$, and M for the intersection $\overline{M} \cap \Lambda_r^\vee$. It follows from [BT, 6.7, 6.9] that Γ_k acts trivially on $\overline{M}$ — hence also on M — because Δ_r is pointwise fixed by the Galois action. (Compare [M 96, 5.2].) Consequently, the loops in M are k-homomorphisms $\mathbb{G}_m \to T$, and they generate a k-split torus S' of rank $|\Delta_r|$ in G.

We define G' to be the derived subgroup of the centralizer in G of S'.

Over k_{sep}, we can describe G' concretely. It follows from (5.3) that G' is generated by the images of the x_α's, where α varies over the roots in Φ whose support does not meet Δ_r. In particular, G' is semisimple and even simply connected by [SS, 5.4b]. The Dynkin diagram of G' is obtained by deleting the vertices Δ_r from the Dynkin diagram of G. The intersection of G' with the maximal torus T is, again over k_{sep}, the image of $\prod_{\alpha_i \notin \Delta_r} h_i$.

We are going to compute $t_{R,G}$ by reducing to this subgroup G'. One reason why this is possible is the following:

Proposition 5.5. *The center Z of G is contained in G'.*

Proof. It suffices to check this over an algebraic closure of k, so we may assume that G' and G are split. It follows from equation (5.3) that the center of G is the subgroup of T cut out by the roots, i.e., it is the intersection

$$Z(G) = \cap_{\alpha \in \Phi} \ker \alpha.$$

Of course, we may replace the condition "$\alpha \in \Phi$" with "α is in the root lattice". In particular, the fundamental weight ω_j corresponding to $\alpha_j \in \Delta_r$ belongs to the root lattice, so $Z(G)$ is contained in the kernel of ω_j. Recall that ω_j is given by the formula

$$\omega_j \left(\prod\nolimits_i h_i(t_i) \right) = t_j.$$

Therefore, $Z(G)$ is a subgroup of $\prod_{\alpha_i \notin \Delta_r} h_i(\mathbb{G}_m)$. But this is the maximal torus of G'. $\square$

5.6. As G' is simply connected, it is of the form $G'_1 \times G'_2 \times \cdots \times G'_s$, where each G'_i is simply connected and absolutely simple with Dynkin diagram a connected component of $\Delta \setminus \Delta_r$.

Lemma. *An element $a \in Z^1(k, \prod_i G'_i)$ can be written as $\prod_i a_i$ for $a_i \in Z^1(k, G'_i)$, and the Rost invariant $r_G(a)$ equals $\sum_i m_i r_{G'_i}(a_i)$, where m_i is the Rost multiplier of the inclusion $G'_i \subset G$.*

Proof. We prove this by induction on the number r of a_i's such that a_i is not the zero cocycle in $H^1(k, G'_i)$. The case $r = 1$ is trivial. We treat the general case. Let i be such that a_i is not zero. Consider the group G_{a_i} obtained by twisting G by

a_i; since G'_i commutes with G'_j for $j \neq i$, the twisted group contains $\prod_{j\neq i} G'_j$ in an obvious way. We find a diagram

$$\begin{tikzcd} H^1(k, \prod_{j\neq i} G_j) \arrow[r] & H^1(k, G_{a_i}) \arrow[r, "r_{G_{a_i}}"] \arrow[d, "\tau_{a_i}"', "\cong"] & H^3(k, \mathbb{Q}/\mathbb{Z}(2)) \arrow[d, "?+r_G(a_i)"] \\ & H^1(k, G) \arrow[r, "r_G"] & H^3(k, \mathbb{Q}/\mathbb{Z}(2)) \end{tikzcd}$$

where τ_{a_i} denotes the twisting map. The diagram commutes as in Remark 2.5. Starting with $\prod_{j\neq i} a_j$ in the upper left, we find $\sum_{j\neq i} m_j r_{G'_j}(a_j)$ in the upper right by the induction hypothesis and a in $H^1(k, G)$ in the lower left. The commutativity of the diagram gives the desired equation. □

5.7. Continue the notation of 5.6. We are mainly interested in the case where G is "simply laced", i.e., where all roots have the same length. In that case, the inclusions $G'_i \subset G$ all have Rost multiplier one—see e.g. [Ga 01b, 2.2] or [M 03, 7.9.2]—and the formula from Lemma 5.6 simply says:

$$r_G(a) = \sum_i r_{G'_i}(a_i). \tag{5.8}$$

6. The center and the torus

For the duration of this section, the field k is arbitrary, with no restriction on the characteristic. We compute how the scheme-theoretic center Z of a semisimple simply connected group G sits inside a fixed maximal torus, in terms of the root system. We do this using an exact sequence from [M 96], which we first recall.

6.1. **Isogenies of tori.** Let $T \to \overline{T}$ be a k-isogeny of tori with kernel Z, i.e., Z is finite and the sequence

$$1 \longrightarrow Z \longrightarrow T \xrightarrow{\pi} \overline{T} \longrightarrow 1$$

is exact. As before, $T_\star$ (resp. $\overline{T}_\star$) is the Γ_k-module of *loops* or cocharacters, i.e., k_{sep} homomorphisms $\mathbb{G}_m \to T_{\mathrm{sep}}$ (resp. $\mathbb{G}_m \to \overline{T}_{\mathrm{sep}}$).

Proposition 6.2. *There is a map z such that*

$$1 \longrightarrow T_\star \xrightarrow{\pi_\star} \overline{T}_\star \xrightarrow{z} \varinjlim_n \mathrm{Hom}_{k_{\mathrm{sep}}}(\boldsymbol{\mu}_n, Z_{\mathrm{sep}}) \longrightarrow 1$$

is an exact sequence of Γ_k-modules.

This is sketched in §1 of Merkurjev's paper (loc.cit.). We give an explicit proof, including a precise description of z which will be used later.

We first prove the following lemma.

Lemma 6.3. *The image of every k-homomorphism from $\boldsymbol{\mu}_n$ to a split torus S is contained in a rank 1 subtorus of S.*

Proof. We may assume that $\boldsymbol{\mu}_n$ injects into S. Dualizing, such an injection corresponds to a surjection $\pi\colon \mathbb{Z}^r \to \mathbb{Z}/n\mathbb{Z}$, where r is the rank of S. By the fundamental theorem for finitely generated abelian groups, there is a basis $b_1, \ldots, b_r$ of $\mathbb{Z}^r$ such that the kernel of π has basis $b_1, b_2, \ldots, b_{r-1}, nb_r$. Projection on the b_r-coordinate defines a map $\pi_r : \mathbb{Z}^r \to \mathbb{Z}$, and π factors through π_r. Dualizing again gives the lemma. □

Proof of Prop. 6.2. Since π has finite kernel, the induced map $\pi_\star$ is injective. We use it to identify $T_\star$ with a submodule of $\overline{T}_\star$. Since the tori have the same rank, the quotient $\overline{T}_\star/T_\star$ is finite.

We now define z. For $\omega \in \overline{T}_\star$, let n be the smallest positive integer such that $n\omega$ belongs to $T_\star$. We define $z_\omega : \boldsymbol{\mu}_n \to T_{\mathrm{sep}}$ to be the restriction of the loop $n\omega \colon \mathbb{G}_m \to T_{\mathrm{sep}}$ to $\boldsymbol{\mu}_n$. Every character χ of $\overline{T}$ maps $\overline{T}_\star \to \mathbb{Z}$, hence $\chi(n\omega)$ is in $n\mathbb{Z}$ and the composition

$$\boldsymbol{\mu}_n \xrightarrow{z_\omega} T \longrightarrow \overline{T} \xrightarrow{\chi} \mathbb{G}_m$$

is 1. Since subgroups of T are cut out by characters, the image of z_ω belongs to Z. It is an exercise to verify that the sequence displayed in the lemma is exact at $\overline{T}_\star$. Note that the map z is Γ_k-equivariant.

It remains to prove that z is surjective. Let $\phi \colon \boldsymbol{\mu}_n \to Z_{\mathrm{sep}}$ be a homomorphism defined over k_{sep}. By Lemma 6.3, there is a rank 1 subtorus S_{sep} of T_{sep} such that the image of ϕ is contained in S_{sep}; let ρ be a loop generating $S_\star$. By hypothesis, $\chi(\pi\rho)$ is in $n\mathbb{Z}$ for every character $\chi \in \overline{T}^\star$, so there is some $\omega \in \overline{T}_\star$ such that $\pi\rho = n\omega$. Then ϕ and z_ω agree in $\varinjlim \mathrm{Hom}(\boldsymbol{\mu}_n, Z_{\mathrm{sep}})$. $\square$

In our situation (with the notations from 5.1 and 5.4), the center Z of G is contained in the maximal torus T and the sequence

$$1 \longrightarrow Z \longrightarrow T \longrightarrow \overline{T} \longrightarrow 1$$

is exact. For the same reason that $T_\star$ is identified with the coroot lattice $\Lambda_r^\vee$, the lattice $\overline{T}_\star$ is identified with $\Lambda^\vee$. Hence the exact sequence given by Prop. 6.2 can be re-written as

$$1 \longrightarrow \Lambda_r^\vee \longrightarrow \Lambda^\vee \longrightarrow \varinjlim_n \mathrm{Hom}_{k_{\mathrm{sep}}}(\boldsymbol{\mu}_n, Z_{\mathrm{sep}}) \longrightarrow 1 \tag{6.4}$$

We close this section with an application of this exact sequence. We will only use it for groups of type B and C, but we include it because of independent interest. Write Δ_c for the set of $\alpha_j \in \Delta$ such that $\breve{\omega}_j$ (as defined in 5.4) is a minuscule weight for the dual root system $\Phi^\vee$. Recall [Bou, §VI.1, Exercise 24] that the minuscule weights are the minimal nonzero dominant weights.

Corollary 6.5. *If G is of inner type and Δ_c does not meet Δ_0, then the center Z of G is contained in the maximal k-split subtorus S of T.*

The condition on Δ_c is the same as saying: every element of Δ_c is circled in the index of G.

Remark 6.6. In the situation of the corollary, $H^1(k, S)$ is zero, so the natural map $H^1_{\mathrm{fppf}}(k, Z) \to H^1(k, G)$ is zero. (If Z is smooth—as in the rest of this paper—the group $H^1_{\mathrm{fppf}}(k, Z)$ agrees with the Galois cohomology group $H^1(k, Z)$.) This has two useful consequences:

(i) The composition (1.1) is zero.
(ii) The connecting homomorphism $(G/Z)(k) \to H^1_{\mathrm{fppf}}(k, Z)$ is surjective.

Proof of Cor. 6.5. Since the minuscule weights generate $\Lambda^\vee/\Lambda_r^\vee$ [Bou, §VI.2, Exercise 5], Z is generated by the images of $z_{\breve{\omega}_j}$ for $\alpha_j \in \Delta_c$. Using the same argument as in the proof of Prop. 6.2, we see that Z is contained in the torus Q (defined over k_{sep}) corresponding to the $n_j\breve{\omega}_j$ for $\alpha_j \in \Delta_c$, where n_j is the smallest natural number such that $n_j\breve{\omega}_j$ is in $\Lambda_r^\vee$.

Since G has inner type, the maximal k-split torus S in T is the intersection of the kernels of $\alpha \in \Delta_0$ by [BT, 6.7, 6.9]. For such an α and for $\alpha_j \in \Delta_c$, we have $\alpha \neq \alpha_j$ by hypothesis, so the inner product $(\alpha, \check{\omega}_j)$ is zero. We conclude that the torus Q is contained in S. □

7. Type B

Before proving the main theorem for exceptional groups, we show that our method gives a new and very short proof for groups of type B_ℓ ($\ell \geq 2$), i.e., that composition (1.1) is zero. By 4.3, it is enough to prove this for groups having Tits index satisfying (4.2). Consulting the tables in [Bou], we find that

$$\Delta_r = \{\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1}\}.$$

The dual root system is C_ℓ, and its only minuscule weight is $\check{\omega}_1$, i.e.,

$$\Delta_c = \{\alpha_1\}.$$

By (4.2), the vertex α_1 is circled in the index of G, so by Remark 6.6(i) the composition (1.1) is zero.

Note that groups of type B_ℓ satisfying the hypotheses of Cor. 6.5 are those of the form $\mathrm{Spin}(q)$ where q is a $(2\ell+1)$-dimensional *isotropic* quadratic form. For these groups, Remark 6.6(ii) says: The spinor norm map $SO(q)(k) \to k^\times/k^{\times 2}$ is surjective.

8. Concrete description of the center

We now give a concrete description of the center Z inside a simply connected group G in terms of the generators and relations for G as in 5.1; consequently we work over the separably closed field k_{sep}. (But see 8.10 below.) Note that, by exactness of sequence (6.4), it is enough to compute $z_{\check{\omega}}$ for $\check{\omega}$ belonging to a set of representatives in $\Lambda^\vee$ of generators of the quotient group $\Lambda^\vee/\Lambda_r^\vee$.

Example 8.1 (E_7)**.** The center of a simply connected group G of type E_7 is isomorphic to $\boldsymbol{\mu}_2$. As all roots have the same length, we can normalize the Weyl-invariant inner product so that all roots have length 2; this identifies Φ with the inverse system $\Phi^\vee$. The fundamental weight

$$\omega_7 = \alpha_1 + \frac{3}{2}\alpha_2 + 2\alpha_3 + 3\alpha_4 + \frac{5}{2}\alpha_5 + 2\alpha_6 + \frac{3}{2}\alpha_7$$

is not in the root lattice and so maps to the nonidentity element in Λ/Λ_r. (We systematically number roots as in Bourbaki; for a diagram see §9.) The corresponding map $z_{\omega_7} \colon \boldsymbol{\mu}_2 \to Z_{\mathrm{sep}}$ is given by

$$z_{\omega_7}(-1) = h_{2\omega_7}(-1) = h_2(-1)h_5(-1)h_7(-1).$$

Here we can see Prop. 5.5 explicitly: $\Delta \setminus \Delta_r$ is $\{\alpha_2, \alpha_5, \alpha_7\}$, and the image of this morphism is contained in G'.

Example 8.2 (E_6)**.** Let now G be of type E_6. The center of G is isomorphic to $\boldsymbol{\mu}_3$. Again we can identify Φ and $\Phi^\vee$. The fundamental weight

$$\omega_1 = \frac{4}{3}\alpha_1 + \alpha_2 + \frac{5}{3}\alpha_3 + 2\alpha_4 + \frac{4}{3}\alpha_5 + \frac{2}{3}\alpha_6$$

is not in in the root lattice; so its class generates Λ/Λ_r. The corresponding isomorphism $z_{\omega_1}\colon \boldsymbol{\mu}_3 \to Z_{\text{sep}}$ is defined by

$$z_{\omega_1}(\zeta) = h_{3\omega_1}(\zeta) = h_1(\zeta)\, h_3(\zeta^2)\, h_5(\zeta)\, h_6(\zeta^2). \tag{8.3}$$

Example 8.4 (C_ℓ)**.** Let now G be of type C_ℓ; the center Z is isomorphic to $\boldsymbol{\mu}_2$. In this case, we must be somewhat more careful because the root system has roots of different lengths. The inverse system $\Phi^\vee$ is B_ℓ. We write $\check{\omega}_\ell$ for the fundamental weight corresponding to the root $\check{\alpha}_\ell$ in the inverse system, i.e.,

$$\check{\omega}_\ell = \frac{1}{2}(\check{\alpha}_1 + 2\check{\alpha}_2 + \cdots + \ell\check{\alpha}_\ell).$$

The corresponding isomorphism $\boldsymbol{\mu}_2 \to Z_{\text{sep}}$ is given by

$$z_{\check{\omega}_\ell}(-1) = \begin{cases} h_{2\check{\omega}_\ell}(-1) = h_1(-1)h_3(-1)\cdots h_{\ell-1}(-1) & \text{if } \ell \text{ is even,} \\ h_{2\check{\omega}_\ell}(-1) = h_1(-1)h_3(-1)\cdots h_\ell(-1) & \text{if } \ell \text{ is odd.} \end{cases} \tag{8.5}$$

Example 8.6 (D_{even})**.** For G of inner type D_ℓ with ℓ even, the center Z is isomorphic to $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$. As for E_7, all roots have the same length and we identify the root system with the inverse system. The fundamental weights

$$\begin{aligned} \omega_{\ell-1} &= \frac{1}{2}\alpha_1 + \alpha_2 + \frac{3}{2}\alpha_3 + \cdots + \frac{\ell-2}{2}\alpha_{\ell-2} + \frac{\ell}{4}\alpha_{\ell-1} + \frac{\ell-2}{4}\alpha_\ell \\ \omega_\ell &= \frac{1}{2}\alpha_1 + \alpha_2 + \frac{3}{2}\alpha_3 + \cdots + \frac{\ell-2}{2}\alpha_{\ell-2} + \frac{\ell-2}{4}\alpha_{\ell-1} + \frac{\ell}{4}\alpha_\ell \end{aligned}$$

have two different residues in Λ/Λ_r. As this is the Klein four-group, they generate it. It follows that the images of the corresponding maps $z_{\omega_{\ell-1}}, z_{\omega_\ell} : \boldsymbol{\mu}_2 \to Z_{\text{sep}}$ generate Z_{sep}. Applying 6.1, we find that Z_{sep} is generated by the images of the homomorphisms z_0, z_1 defined by

$$\begin{aligned} z_0(-1) &:= h_1(-1)h_3(-1)\cdots h_{\ell-3}(-1)\, h_\ell(-1) \\ z_1(-1) &:= h_1(-1)h_3(-1)\cdots h_{\ell-3}(-1)\, h_{\ell-1}(-1) \end{aligned} \tag{8.7}$$

Note that we have abandoned tying the weights with their corresponding homomorphism, because this depends on the parity of $\ell/2$.

Example 8.8 (A_2)**.** Suppose now that G has type A_2, i.e., G is isomorphic to SL_3 and its center is isomorphic to $\boldsymbol{\mu}_3$. Since the fundamental weight $\omega_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$ does not belong to the root lattice, we get an isomorphism $z_{\omega_1} : \boldsymbol{\mu}_3 \to Z_{\text{sep}}$ defined by

$$z_{\omega_1}(\zeta) = h_1(\zeta^2)h_2(\zeta). \tag{8.9}$$

8.10. Let G be a simply connected semisimple group of inner type over k. Each element $\check{\omega}$ of $\Lambda^\vee$ defines an isomorphism

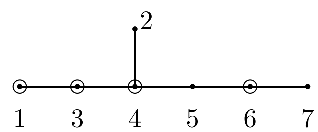
$$z_{\check{\omega}}\colon \boldsymbol{\mu}_n \to \boldsymbol{\mu}_n \subset G$$

over k_{sep}. But the only morphisms $\boldsymbol{\mu}_n \to \boldsymbol{\mu}_n$ are raising to a power, so $z_{\check{\omega}}$ is Γ_k-equivariant, and hence is defined over k by [Borel, AG.14.3].

9. Type E_7

This section proves the main theorem 2.3 for a group G of type E_7. In this case, the center Z is identified with $\boldsymbol{\mu}_2$. We prove that $t_{R,G}$ coincides with the Tits class t_G, i.e. the composition (1.1) is the cup product with t_G, where the cup product is induced by the obvious bilinear map $Z(k_{\text{sep}}) \times Z(k_{\text{sep}}) = \boldsymbol{\mu}_2(k_{\text{sep}}) \times \boldsymbol{\mu}_2(k_{\text{sep}}) \to \boldsymbol{\mu}_2(k_{\text{sep}})^{\otimes 2}$. (The group $H^2(k, Z)$ is 2-torsion, so $t_{R,G}$ and t_G generate the same subgroup of $H^2(k, Z)$ if and only if they are equal.)

By 4.3, it is enough to prove this for groups of type E_7 with Tits index satisfying (4.2). Let G be such a group, i.e. assume the index of G is



or possibly has more circled vertices. (If the index has more circles, then G is split, see [Ti 66, p. 57].)

Write G'_i with $i = 2, 5, 7$ for the components of G', where G'_i corresponds to the vertex α_i in the Dynkin diagram. Each is isomorphic to $SL_1(Q_i)$ for some quaternion algebra Q_i. Since the weights $\omega_2, \omega_5, \omega_7$ all have the same image in Λ/Λ_r, the Q_i are all isomorphic by [Ti 71, p. 211] and we simply write Q. By [Ti 71, loc. cit.], $[Q] \in H^2(k, \boldsymbol{\mu}_2)$ is also the Tits class of G. The center Z is contained in the product $Z_2 \times Z_5 \times Z_7$ of the centers of G'_2, G'_5, G'_7 by Prop. 5.5. By the description of the center given in example 8.1, the map

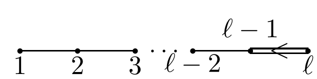
$$\boldsymbol{\mu}_2 = Z \to Z_2 \times Z_5 \times Z_7 = \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$$

is $-1 \mapsto (-1, -1, -1)$. (Even though Example 8.1 treats G over the separable closure, it still applies to our nonsplit G over k by 8.10.) Applying equation 5.8 and the type A_1 case of the main theorem (see Remark 2.4), we get that the class $t_{R,G}$ is $3[Q] = [Q]$, which is the Tits class of G.

This concludes the proof of the main theorem for groups of type E_7.

10. Type C

The argument of the previous section can be adapted to recover the theorem for groups of type C_ℓ. (Alternatively, in the case where ℓ even, the same argument as for type B in §7 shows that composition (1.1) is zero.) Since the result is not new for type C, we only briefly sketch the proof. We want to show that $t_{R,G}$ is 0 if ℓ is even and t_G if ℓ is odd. Recall that a group of type C_ℓ has Dynkin diagram



As before, it is enough to prove it for groups whose Tits index satisfy condition (4.2), i.e., such that α_i is circled for i even.

As before, we write G'_i for the component of G' corresponding to the vertex α_i, for $1 \leq i \leq n$, i odd. We may identify all those with $SL(Q)$ for some quaternion

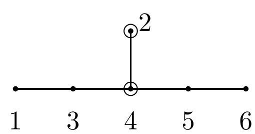
algebra Q. The description of the center given in Example 8.4 shows that Z maps to the product of the centers of the G'_i by $(-1) \mapsto (-1, \dots, -1)$.

It remains to apply Lemma 5.6, which asserts $r_G(a) = \sum_i m_i r_{G'_i}(a_i)$. Note that we do have to care about Rost multipliers here, since groups of type C are not simply laced. If $1 \le i \le n-1$, i odd, then the root α_i is short in C_ℓ. So the corresponding co-root $\check{\alpha}_i$ is long and the inclusion $G'_i \subset G$ has Rost multiplier 2, see e.g. [Ga 01b, 2.2] or [M 03, 7.9.2]. Since the Rost invariant has values in $H^3(k, \boldsymbol{\mu}_2^{\otimes 2})$ which is 2-torsion, we get $r_G(a) = 0$ if ℓ is even, and $r_G(a) = r_{G'_\ell}(a_\ell)$ if ℓ is odd. In this last case, applying the formula for groups of type A, we get that $t_{R,G} = [Q]$, which is the Tits class of G.

11. Type E_6

In this section, we prove the theorem for a group G of type E_6.

11.1. Groups of inner type E_6. Let us first assume G is of inner type, that is Γ_k acts trivially on the Dynkin diagram. The center Z is identified with $\boldsymbol{\mu}_3$ and we want to prove that $t_{R,G}$ and t_G generate the same subgroup of $H^2(k, Z)$. Again by 4.3, it suffices to consider groups whose Tits index satisfies condition (4.2). Let G be such a group, i.e. we assume the Tits index of G is



or has more circled vertices. (As in the type E_7 case in §9, if more vertices are circled, then G is split.)

Write G'_1 and G'_5 for the components of G' corresponding to the subdiagrams with vertices 1, 3 and 5, 6 respectively. They are of type A_2. Moreover, since the fundamental weights ω_1 and ω_5 have the same class in Λ/Λ_r, you may identify G'_1 and G'_5 with $SL(D)$ for some central simple algebra D of degree 3 over k. The center of $SL(D)$ is naturally identified with $\boldsymbol{\mu}_3$, and the composition

$$H^1(k, \boldsymbol{\mu}_3) \to H^1(k, SL(D)) = H^1(k, G'_i) \xrightarrow{r_{G'_i}} H^3(k, \mathbb{Q}/\mathbb{Z}(2))$$

is the cup product with $m[D]$ for $m = 1$ or 2 by Example 2.4.

Comparing the description of the center of G and the G'_i given by formulas 8.3 and 8.9, one sees that the map $Z \to G'_1 \times G'_5$ is $\zeta \mapsto (\zeta^2, \zeta^2)$. Hence the induced map $H^1(k, Z) \to H^1(k, Z'_1) \times H^2(k, Z'_5)$ is given by $a \mapsto (2a, 2a)$. Applying Equation (5.8), we see that the composition (1.1) is

$$a \mapsto 4a \cdot m[D] \quad \in H^3(k, \boldsymbol{\mu}_3^{\otimes 2}).$$

But $4a$ equals a because the exponent of Z is 3. Therefore, $t_{R,G}$ equals $m[D]$, i.e., generates the same subgroup of $H^2(k, Z)$ as $[D]$, which is t_G [Ti 71, p. 211].

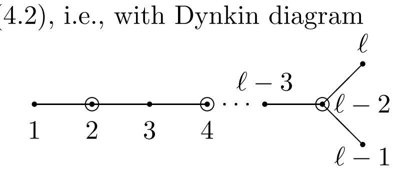
11.2. Groups of outer E_6 type. Suppose now that the group G is of outer type, i.e., the absolute Galois group acts nontrivially on the Dynkin diagram of G. The kernel of the homomorphism $\Gamma_k \to \mathrm{Aut}(\Delta) = \mathbb{Z}/2$ defines a quadratic extension L of k, over which the group becomes of inner type. Its center Z is isomorphic to $\boldsymbol{\mu}_{3[L]}$.

By the inner type case, we know that $t_{R,G} - m\, t_G$ is killed by L. Since it lives in the group $H^2(k, Z)$ which is of exponent 3, while L is quadratic over k, a restriction/corestriction argument shows that $t_{R,G} - m\, t_G$ is zero in $H^2(k, Z)$. This concludes the proof for groups of type E_6.

12. Type D_ℓ for ℓ even

We now prove the main theorem for groups of type D_ℓ for ℓ even, including trialitarian D_4.

12.1. Groups of inner type D_ℓ. As before, we first assume that G is of inner type, so that Z is identified with $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$. We want to prove that the composition (1.1) is the cup product with the Tits class, where the cup product is induced by the bilinear map given in (2.2). By 4.3, it is enough to consider groups with Tits index satisfying condition (4.2), i.e., with Dynkin diagram



or having more circled vertices.

We write G'_i, for $i = 1, 3, \ldots, \ell-3, \ell-1$ and ℓ for the component of G' corresponding to the vertex α_i. They are of type A_1, and may be identified with $SL(Q_i)$, for some quaternion algebras Q_i. Each has center Z'_i isomorphic to $\boldsymbol{\mu}_2$. Moreover, since $\omega_1, \ldots, \omega_{\ell-3}$ and $\omega_{\ell-1} + \omega_\ell$ have the same class in Λ/Λ_r, we may assume $Q_1 = \cdots = Q_{\ell-3} = Q$, for some quaternion algebra Q satisfying $[Q] = [Q_{\ell-1}] + [Q_\ell]$.

Since G is of inner type, the homomorphisms z_0, z_1 from Example 8.6 are k-defined even though G need not be split. Hence, we get a map

$$(z_0, z_1)\colon \mu_2 \times \mu_2 \to Z \subset Z'_1 \times \cdots \times Z'_{\ell-3} \times Z'_{\ell-1} \times Z'_\ell.$$

The induced map $H^1(k, Z) \to \prod_i H^1(k, Z'_i)$ is given by

$$(a_0, a_1) \mapsto (a_0 + a_1, \ldots, a_0 + a_1, a_1, a_0).$$

Applying Equation (5.8), and the formula for groups of type A (see 2.4), we get $r_G(a_0, a_1) = \frac{\ell-2}{2}(a_0 + a_1)\cdot[Q] + a_1\cdot[Q_{\ell-1}] + a_0\cdot[Q_\ell]$, that is

$$r_G(a_0, a_1) = \begin{cases} a_0\cdot[Q_{\ell-1}] + a_1\cdot[Q_\ell] & \text{if } \ell \equiv 0 \pmod 4 \\ a_0\cdot[Q_\ell] + a_1\cdot[Q_{\ell-1}] & \text{if } \ell \equiv 2 \pmod 4 \end{cases} \tag{12.2}$$

On the other hand, we have:

$$\omega_{\ell-i}(a_0, a_1) = a_i \quad \text{and} \quad \omega_{\ell-i}(t_G) = [Q_{\ell-i}] \tag{12.3}$$

for $i = 0, 1$. Combining (12.2) and (12.3) with the definition of the cup product in (2.2), we conclude that composition (1.1) equals the cup product with the Tits class t_G.

12.4. Groups of outer D_ℓ type. For the groups of type ${}^2D_\ell$, one can reduce to the ${}^1D_\ell$ case as in [MPT, p. 817]. Alternatively, one can apply the same method as for groups of inner type. In this case, G' is a direct product of $(\ell-2)/2$ copies of $SL(Q)$ and a transfer $R_{L/k}SL(Q')$, where Q' is a quaternion algebra over a quadratic extension L of k. To apply (5.8), one needs to know the Rost invariant of $R_{L/k}SL(Q')$, which is specified in [M 03, 9.8].

It remains only to treat the trialitarian groups, i.e., the groups G of type ${}^3D_4$ or ${}^6D_4$. Recall what this means. The automorphism group of the Dynkin diagram Δ of G is the symmetric group on three letters, and the superscript 3 or 6 denotes the size of the image of homomorphism $\Gamma_k \to \mathrm{Aut}(\Delta)$. The kernel of this homomorphism fixes at least one separable cubic field extension of k; we pick one and call it L. The group G is of type ${}^1D_4$ or ${}^2D_4$ over L. As the exponent of Z (i.e., 2) is relatively prime to the dimension of L/k (i.e., 3), the proof may be completed using some restriction/corestriction argument as for groups of outer type E_6.

13. Application to groups of type E_7

We now apply the main theorem to prove a result about groups of type E_7. A simply connected group of type E_7 can be described in terms of "gifts" as in [Ga 01a]. A gift is a triple (A, σ, π) where A is a central simple k-algebra of degree 56, σ is a symplectic involution on A, and $\pi\colon A \to A$ is k-linear and satisfies certain axioms, see [Ga 01a, Def. 3.2] for a precise statement. The Brauer class of A is the Tits class t_G. The k-points of G are the elements $a \in A^\times$ such that $\sigma(a)a = 1$ and $\mathrm{Int}(a)\,\pi = \pi\,\mathrm{Int}(a)$, where $\mathrm{Int}(a)$ denotes the automorphism $x \mapsto axa^{-1}$ of A.

The associated adjoint group $\overline{G}$ is a subgroup of $PGL(A)$; it has k-points $\mathrm{Int}(a)$ for $a \in A^\times$ such that $\sigma(a)a$ is in $k^\times$ and $\mathrm{Int}(a)\,\pi = \pi\,\mathrm{Int}(a)$. The exact sequence

$$1 \longrightarrow \boldsymbol{\mu}_2 \longrightarrow G \longrightarrow \overline{G} \longrightarrow 1$$

gives a connecting homomorphism

$$\delta\colon \overline{G}(k) \to H^1(k, \boldsymbol{\mu}_2) = k^\times/k^{\times 2}.$$

As in [KMRT, §31], one finds

$$\delta(\mathrm{Int}(a)) = \sigma(a)a \in k^\times/k^{\times 2}.$$

If $\lambda k^{\times 2}$ is in the image of δ, then its image in $H^1(k, G)$ is zero and the composition (1.1) sends λ to zero. The main theorem gives:

Corollary 13.1. *If $\lambda k^{\times 2}$ is in the image of δ, then $(\lambda)\cdot[A] = 0$.*

This result was previously observed in the case $k = \mathbb{R}$ in [Ga 01a, 6.2].

Acknowledgements. The first author thanks Université Paris 13 for its hospitality while some of the work on this paper was performed.

References

[Borel] A. Borel, *Linear algebraic groups*, second ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991.

[Bou] N. Bourbaki, *Lie groups and Lie algebras: Chapters 4–6*, Springer-Verlag, Berlin, 2002.

[BS] A. Borel and T.A. Springer, *Rationality properties of linear algebraic groups. II*, Tohoku Math. J. (2) **20** (1968), 443–497, (= Borel, Oe., vol. 2, #76).

[BT] A. Borel and J. Tits, *Groupes réductifs*, Inst. Hautes Études Sci. Publ. Math. **27** (1965), 55–150.

[Ga 01a] R.S. Garibaldi, *Groups of type E_7 over arbitrary fields*, Comm. Algebra **29** (2001), no. 6, 2689–2710, [DOI 10.1081/AGB-100002415].

[Ga 01b] ———, *The Rost invariant has trivial kernel for quasi-split groups of low rank*, Comment. Math. Helv. **76** (2001), no. 4, 684–711.

[Gi 00] Ph. Gille, *Invariants cohomologiques de Rost en caractéristique positive*, K-Theory **21** (2000), 57–100.

[KMRT] M.-A. Knus, A.S. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, Colloquium Publications, vol. 44, Amer. Math. Soc., 1998.

[M 96] A.S. Merkurjev, *A norm principle for algebraic groups*, St. Petersburg Math. J. **7** (1996), no. 2, 243–264, [Russian original: Algebra i Analiz **7** (1995) 77–105].

[M 03] ———, *Rost invariants of simply connected algebraic groups*, Cohomological invariants in Galois cohomology, University Lecture Series, vol. 28, Amer. Math. Soc., 2003, with a section by S. Garibaldi.

[MPT] A.S. Merkurjev, R. Parimala, and J.-P. Tignol, *Invariants of quasi-trivial tori and the Rost invariant*, St. Petersburg Math. J. **14** (2003), 791–821.

[MT] A.S. Merkurjev and J.-P. Tignol, *The multipliers of similitudes and the Brauer group of homogeneous varieties*, J. Reine Angew. Math. **461** (1995), 13–47.

[SS] T.A. Springer and R. Steinberg, *Conjugacy classes*, Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69), Springer, Berlin, 1970, pp. 167–266.

[St] R. Steinberg, *Lectures on Chevalley groups*, Yale University, New Haven, Conn., 1968.

[Ti 66] J. Tits, *Classification of algebraic semisimple groups*, Algebraic Groups and Discontinuous Subgroups, Proc. Symp. Pure Math., vol. IX, AMS, 1966, pp. 32–62.

[Ti 71] ———, *Représentations linéaires irréducibles d'un groupe réductif sur un corps quelconque*, J. Reine Angew. Math. **247** (1971), 196–220.

Department of Mathematics & Computer Science, Emory University, Atlanta, GA 30322, USA

E-mail address: skip@member.ams.org

URL: http://www.mathcs.emory.edu/~skip/

Laboratoire Analyse, Géométrie & Applications, UMR CNRS 7539 – Institut Galilée, Université Paris 13, 93430 Villetaneuse, France

E-mail address: queguin@math.univ-paris13.fr

URL: http://www-math.univ-paris13.fr/~queguin/